ON SOPHUS LIE'S REPRESENTATION OF IMAGINARIES IN PLANE GEOMETRY.

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The first published paper of Sophus Lie appeared in the Transactions of the Academy of Christiania in February, 1869, and was concerned with the subject of this article.* So great a rôle does this memoir play in the scientific career of the author that a reference to it in the *Leipziger Berichte*, 1897, p. 726, elicits from him the observation that it formed the starting point of all his mathematical investigations.† In fact, the point of view there adopted leads in the most natural way to a point-line transformation of space which specialized gives the celebrated line-sphere transformation,‡ and generalized leads to a new duality defined by two *aequationes directrices* in two sets of point coordinates, upon which is based the monumental memoir in the fifth volume of the *Mathematische Annalen* (1872) : Ueber Complexe, insbesondere Linien- und Kugel-Complexe, etc.

I propose to give a presentation of some of the results in the paper referred to above, and I am led to do this not only for the reason that the original lacks clearness and continuity, but also from the fact that Lie's work has been apparently overlooked by writers on this subject.§ Moreover, I think the method to be expounded possesses merits not shared by such other representations as I have seen. The essential difference consists in this, that Lie assumes the straight line as geometric element in the plane, while other representations are based upon point geometry. Thus the latter give a real line in space as image of a real or imaginary point in the plane,‖ while Lie's method also leads

* The full title is : Repräsentation der Imaginären der Plangeometrie. (Jeder plangeometrischer Satz ist ein besonderer Fall eines stereometrischen Doppel-Satzes in der Geometrie der Linien-Congruenzen). The paper appeared in two parts, pages 16–38, 107–146, the first part, with which we are especially concerned, having previously appeared in *Crelle*, vol. 70 (1869), p. 346. My references are to the Transactions.

† *Cf.* also Lie-Scheffers, *Geometrie der Berührungstransformationen*, vol. 1, p. 326, 448.

‡ First published in the note, Sur une transformation géométrique, *Comptes Rendus*, vol. 71 (1870), p. 579.

§ *Cf.* Coolidge, A purely geometric representation of all points in the projective plane, *Transactions Amer. Math. Soc.*, vol. 1 (1900), p. 182.

‖ For a very simple representation *cf.* the memoir by Duport, Sur un mode particulier de représentation des imaginaires, *Annales de l'école normale*, 1880, p. 301.

to a real line in space as representative of a real or imaginary *line* in the plane, each real element moreover being its *own* image. In Lie's representation there is then no change in *kind*.

To all such representations of the imaginaries in Plane Geometry there are of course two objections: *first*, that generalization to three dimensions is futile, and *secondly*, that the content of propositions in the plane is so greatly changed as to be of questionable value. The results of Lie's paper successfully meet the latter objection, I think; while perhaps all that can be said to justify a system such as von Staudt's, which substitutes a real substratum for the imaginaries in three dimensions, is summed up in Lindemann's opinion* that the *recognition of the possibility* of such a representation is a great advantage.† It is, however, right to add that von Staudt's work has led to a well developed theory of involutions.

As to the present presentation as compared with the original, I would say that the definition which I have adopted in §1 is purely projective, and the discussion proceeds in the simplest possible manner therefrom. From this the representation of Lie is deduced in §2 as a metrical special case. The latter, however, in his paper, makes use of an analytic definition only. He there gives, in fact, three different methods of representation, all defined analytically, and it is with the second of these (p. 32 *l. c.*) that we have to do. This is in substance identical with the first, as Lie points out, while the third method is merely defined and not developed. In the second part of the paper referred to, Lie develops synthetically the natural consequences of the representation.

We shall have to deal in our discussion with the four species of imaginary elements in space which present themselves in von Staudt's system, or also by allowing the equations of Analytic Geometry to contain complex coefficients. These are (1) *imaginary points*, each lying on one real line, (2) *imaginary planes*, each containing one real line, (3) *imaginary lines of the first kind*, each lying in one real plane and containing one real point, (4) *imaginary lines of the second kind*, possessing neither a real plane nor a real point.

Since the representation introduces the right line as element of space, we shall have to do with some of the notions of line geometry.‡ Thus in §1

* Clebsch-Lindemann: *Vorlesungen über Geometrie*, vol. 2, p. 130. An excellent presentation of von Staudt's theory is given in this volume, page 104.

† Compare also Professor Scott's critique: The status of imaginaries in pure geometry, *Bull. Amer. Math. Soc.*, ser. 2, vol. 7 (1900), p. 163.

‡ Koenigs, *La géométrie reglée.* Plücker, *Neue Geometrie des Raumes, etc.* Clebsch-Lindemann, *l. c.* p. 41.

we are led to the configuration of all lines of space intersecting two fixed lines, or, in the language of line geometry, to a linear line congruence of which the fixed lines are the directrices. The middle point of the common perpendicular of the directrices is the *centre*, and one-half of this shortest distance the *constant* of the congruence. I may refer to the text-books quoted for further details.

In von Staudt's system a linear line congruence is the real representative of an imaginary line of the second kind, the congruence consisting of all lines intersecting this line and the conjugate imaginary line.

1. Geometrical Definition of the Representation in Projective Form. Let σ be a real plane in space, the *fundamental plane*, and I an imaginary point, the *fundamental point*, not in σ. Also let L_0, the *fundamental line*, be the real line of the point I, and let S_0, the *fundamental pencil*, be the pencil of real lines in σ whose centre p_0 is the *foot* of the line L_0, that is the point in which L_0 meets σ.

Then any line l in σ may be represented by a real line L in space, viz., the real line of the plane containing I and l.

The line L is uniquely determined save when l belongs to the fundamental pencil, for in that case and in that case only is the plane of I and l real, containing as it does two real lines L_0 and l, and accordingly *L is now any line in a real plane containing the fundamental line.* If l is any other real line in σ, then L will coincide with l.

Conversely, every real line L in space is represented by a line l of σ, viz. the intersection of σ with the plane containing I and L.

The line l is uniquely determined save when L coincides with the fundamental line L_0, for then the plane of I and L becomes indeterminate and l is any line in σ through p_0.

We have therefore established in this way a (1, 1) correspondence between all the lines of σ and the real lines of space, the uniqueness failing only for the lines of the fundamental pencil in σ, and for the fundamental line in space.

It should be noted that l and L intersect in the real point on l.

Consider now in σ the locus of the first class, a point p. The ∞^2 lines of σ through p are represented by the real lines of ∞^2 planes passing through the line Ip. Each of these ∞^2 real lines, however, intersects not only Ip but also its conjugate line $J\bar{p}$, where J and $\bar{p}$ are the conjugate points of I and p respectively. Hence:

Any point p in σ is represented by a real linear line congruence whose directrices are the conjugate lines Ip and $J\bar{p}$.

The line of the congruence lying in σ is the real line through p. It should be remarked also that in general Ip and $J\bar{p}$ are imaginary lines of the second kind.

The congruence becomes special, *i. e.* the directrices intersect, when and only when the real line through p belongs to the fundamental pencil. For, since $\bar{p}$ also lies on this line, Ip and $J\bar{p}$ are now coplanar and accordingly intersect in a real point P. The congruence then consists of all lines through P and all lines in the plane of Ip and $J\bar{p}$, *i. e.* the plane containing P and the fundamental line L_0. Therefore:

A point in σ whose real line belongs to the fundamental pencil is represented by a special congruence consisting of all lines passing through a real point and all lines in the plane determined by that point and the fundamental line.

If p is real, then P coincides with it.

This association at the ∞^3 points of σ, whose real lines belong to the fundamental pencil, and the ∞^3 real points of space is very striking and of great importance in the sequel. The point p_0 in σ is represented by all lines intersecting the fundamental line L_0, *i. e.* by a special linear line complex whose axis is the fundamental line. This complex may be called the *fundamental complex*. The fundamental line L_0 belongs to every congruence representing a point p of σ, for it corresponds to the line $p\,p_0$.

Consider, now, a pencil of ∞^1 lines in σ through a point p, determined, for example, by a real parameter. Then we have the following:

THEOREM I. *The ∞^1 lines of a pencil in σ whose centre is p are represented in space by the generators of one system of a ruled quadric passing through I and J.*

Proof. The real line in space which represents a line of the given pencil in σ is the intersection of conjugate planes passing respectively through Ip and the conjugate line $J\bar{p}$.

Since the first plane intersects σ in a line of the given pencil, the ∞^1 given lines in σ evidently give rise to two projective pencils of planes through Ip and $J\bar{p}$, respectively, and, by a well known theorem, the line of intersection of corresponding planes of two projective pencils generates a ruled quadric. Finally, Ip and $J\bar{p}$ evidently belong to the generators of the other system, and therefore the quadric passes through I and J.

The quadric degenerates into two flat pencils when the real line through p belongs to the fundamental pencil and to the pencil considered. For, as explained above, in this case the congruence representing p becomes special.

Consider, next, a curve in σ of the second class, a conic C_2. This gives rise in space to a line congruence* of the second order; for through any point P in space will pass two lines of the congruence representing the tangents in σ to C_2 drawn through the foot of IP.

The cone with vertex I passing through C_2 is one of the focal surfaces of this congruence; for if consecutive lines of the latter intersect, then the corresponding consecutive tangents to C_2 do likewise, and this intersection is a point p on C_2; therefore the two lines of the congruence belong to a linear congruence and can intersect only on Ip. The entire focal surface consists of the above cone and the corresponding conjugate cone with vertex at J. The line congruence is then of the second order and fourth class. The real intersection of the focal surfaces, in general a skew curve of order four, corresponds to the points common to C_2 and the fundamental pencil. The tangents of this curve belong to the congruence.

If the conic C_2 passes through the foot p_0 of the fundamental line, then the real intersection of the focal surfaces degenerates into L_0 and a skew curve of the third order passing through I and J. Finally, if the tangents to C_2 through p_0 are real, then it is readily seen that the above cones have double contact on the line which represents the chord of contact of these tangents, and therefore their intersection consists of two conics.

Without prolonging the discussion let it suffice to state that a curve of class n in σ becomes in space a line congruence of order n whose focal surfaces are conjugate cones with vertices I and J. The tangents of the real intersection of these cones belong to the congruence.

2. Lie's Analytical Definition as a Metrical Special Case. Taking XZ as fundamental plane σ, and one of the circular points in the XY plane as I we obtain (using rectangular coordinates) Lie's definition. In fact, if

$$cx = bz + a \qquad (c \text{ real})$$

is any line l in XZ, then the plane through I and l is

$$c(x + iy) = bz + a,$$

and if

$$b = m + in, \qquad a = p + iq,$$

* In this connection the reader is referred to Picard, *Traité d'analyse*, vol. 1, p. 301.

the real line L of this plane is

$$cx = mz + p, \qquad cy = nz + q.$$

Conversely, given the equations of any real line L of space in this form, we obtain the equation of the corresponding line l in XZ by multiplying the second of these equations by i, adding, and then placing $y = 0$. The plane

$$c(x + iy) = bz + a$$

is real when and only when $c = o$ and b/a is real. Thus we see that the fundamental pencil consists of all real lines in XZ parallel to XX'. The line at infinity in XY is, of course, the fundamental line, as is evidenced by the obvious fact that for $c = o$ the equations of L reduce to those of the line in question, while l becomes $z =$ any constant. J is the other circular point in XY, and p_0 the point at infinity on XX'.

Any line in σ passing through a given point p (α, β) is given by

$$c(x - \alpha) = b(z - \beta),$$

where c and b are arbitrary constants, c being taken real. The plane containing this line and I is then:

(1) $$c(x + iy - \alpha) = b(z - \beta).$$

This equation defines ∞^2 planes (b complex) through the line Ip

(2) $$x + iy = \alpha, \quad z = \beta.$$

Let $\alpha = a_1 + i\alpha_2$, $\beta = \beta_1 + i\beta_2$. Now if $\beta_2 = 0$, *i. e.* if the real line through p belongs to the fundamental pencil, then the line (2) contains the real point $P(\alpha_1, \alpha_2, \beta_1)$ and is an imaginary line of the first kind. The real lines of the planes (1), if $c \neq 0$, pass through P; but for $c = 0$ the plane (1) becomes real, and we get all lines of the real horizontal plane through P, $z = \beta_1$.

If however $\beta_2 \neq 0$, (2) is always an imaginary line of the second kind, and the real lines of the ∞^2 planes (1) intersect (2) and also the conjugate line

(3) $$x - iy = a_1 - i\alpha_2, \qquad z = \beta_1 - i\beta_2.$$

The lines (2) and (3) are accordingly the directrices of the linear congruence formed by all real lines in the planes (1). The shortest distance between them is $2i\beta_2$, and the middle point of this line is (α_1, a_2, β_1), so that we have, adopting the nomenclature of Plücker,

THEOREM 2. *In Lie's analytic representation* the imaginary point* $(\alpha_1 + i\alpha_2, \beta_1 + i\beta_2)$ *goes over into the* ∞^2 *lines of a linear line congruence whose centre is* $(\alpha_1, \alpha_2, \beta_1)$ *and constant* $i\beta_2$.

The first method given by Lie in the paper referred to represents the imaginary point $(\alpha_1 + i\alpha_2, \beta_1 + i\beta_2)$ by the real point of space $(\alpha_1, \alpha_2, \beta_1)$, to which is assigned a *weight* β_2. He very soon points out (page 20) that β_2 measures the constant of a line congruence, but adheres by preference to this representation throughout the paper. The ∞^3 points of the fundamental plane for which $\beta_2 = 0$, *i. e.* the points whose real lines constitute the fundamental pencil, he calls *null points*, which name I shall adopt. The important point is this, that each null point is represented by a special congruence whose centre is a real point P and whose plane is the plane PL_0. We may say, indeed, that the image of the null point is P itself, and conversely the special congruence constructed as aforesaid with P leads back to the original null point. Ambiguity enters only for the point p_0 which gives any point in the fundamental line.

THEOREM 3. *Lie's representation establishes a* $(1, 1)$ *involutory correspondence between the lines of the plane and the real lines of space in which the fundamental configurations are, in the plane a pencil of real lines, and in space a line through the vertex of this pencil. By this is also set up a* $(1, 1)$ *involutory correspondence of the* ∞^3 *points on the lines of this pencil (the null points of the plane), and the real points of space, and the uniqueness fails only for the vertex of the pencil.*

In this theorem are set forth the peculiar merits of Lie's method to which I have already referred.

It is to be observed that the combination of null point p and line containing p is represented by a real point P and a real line through P. Also the range of null points on any line l is projective with the real range on the corresponding line in space.

3. Correlation in the Plane and the Corresponding Point-Line Transformation of Space. On page 35 of Lie's paper is introduced the question of what arises in space when the representation is carried out on a correlation in the plane. The answer is immediate, for since the ∞^3 nul-points of σ go over into ∞^3 lines, then we obviously have a transformation T of the ∞^3 points of space into an assemblage of ∞^3 lines, *i. e.* a line complex

* *Lie*, *l. c.* p. 33.

Γ. The general features of T we may deduce by considering a polar reciprocation in σ with respect to a conic C_2 (Lie, §27). Since if p in σ reciprocates into l, then any null point of l corresponds to a line through p, we may state one fundamental property of T thus:

THEOREM 4. *If a point P in space describes a line L of the complex Γ, the corresponding line will turn around the point corresponding to L, i. e. will generate a cone whose vertex is that point.*

We next determine the degree of this cone (which we shall call a *complex cone*), which will give us the *order* of the line complex. If P describes an arbitrary line L the corresponding ∞^1 lines of Γ will represent the ∞^1 lines of a pencil in σ, *i. e.* by Theorem 1, will be generators of one system of a ruled quadric through I and J. Hence

THEOREM 5. *The complex cones of Γ are quadric cones through I and J, and therefore Γ is a line complex of order* 2.

Consideration of the relation of C_2 to the fundamental pencil in σ will lead to the singular properties of Γ. In the general case, p_0 reciprocates into a line l_0 which corresponds to a line L_0' of Γ. Then, as P describes L_0', the corresponding complex cone must degenerate into two planes; for the vertex must correspond to the base point p_0, *i. e.*, is any point of IJ. We assume this plane-pair *real*,* say E_1 and E_2, intersecting σ in l_1 and l_2. Now the ∞^3 lines of Γ are grouped in ∞^1 linear congruences representing the range on l_0 into which the lines of the fundamental pencil reciprocate. Two points of this range are null points, viz. the intersections of l_0 with l_1 and l_2; hence two *special* congruences belong to Γ; their centres are the intersections of L_0' with E_1 and E_2, and their planes are E_1 and E_2. If A_1, A_2 are the points of intersection of L_0' with E_1, E_2, then A_1 transforms into any line in E_2 while any point in E_2 gives a line through A_1. And since an arbitrary line L in space intersects E_1 and E_2, as P describes L the ruled quadric generated by the corresponding complex line passes through A_1 and A_2. The complex cones then also contain these points. We then easily obtain the following definition of Γ:

Given the tetrahedron A_1A_2IJ and a complex line L, then Γ consists of the generators of the ∞^2 ruled quadrics circumscribed to the tetrahedron and containing L, which belong to the same system as L.

For if p is the pole of l, and l' any line through p, then the ∞^1 null points

* This assumption is made in order to use our representation.

on l' give the generators of such a quadric; and all lines of Γ are determined by taking for l' any line through p. Since the definition is symmetrical with respect to the vertices of the tetrahedron, we see that every line through each vertex and in each face of the tetrahedron belongs to Γ. The complex Γ is then known to be a *tetrahedral complex.** The complex lines intersect the faces of the tetrahedron $A_1 A_2 IJ$ in a constant cross ratio.

THEOREM 6. *The general polar reciprocation in the fundamental plane yields, under Lie's representation, a transformation of the ∞^3 points of space into the lines of a tetrahedral complex, the points I and J being vertices of the fundamental tetrahedron.*†

The complex Γ will degenerate and the transformation take on special properties if we assume a special relation of C_2 and the fundamental pencil. For example, let the point p_0 be on C_2. Then it is readily seen that the complex has two singular points I, J and two singular planes intersecting in IJ. These planes intersect σ in the tangent to C_2 at p_0 and in the conjugate line. Without discussing the corresponding transformation, consider next what arises if p_0 reciprocates into a line l_0 belonging to the fundamental pencil. Then Γ degenerates into two special linear complexes, whose axes are the line IJ and a line in the plane IJl_0. The corresponding transformation of space becomes very simple. This case, however, as well as others of particular interest, appear if we do not limit ourselves to a reciprocation.‡

Consider then a correlation in σ such that p_0 transforms into a line l_0 of the fundamental pencil. Then a second line l_0' (in general imaginary) through p_0 will transform back into p_0. In the first instance the fundamental pencil correlates into a range on l_0 not including p_0, and the corresponding complex in space consists of ∞^1 special linear congruences whose centres lie upon a non-degenerate conic through I and J in the plane of IJ and l_0. That is, *the complex Γ is now a special quadratic complex consisting of all secants of a conic through I and J.* On the contrary the range determined on l_0' dual with the fundamental pencil contains one null point p_0, and accordingly the complex arising in space degenerates into two linear complexes one of which is

* The reader is referred to Lie-Scheffers, *l. c.* p. 311, for an excellent discussion of this complex.

† This transformation was first remarked by Reye in 1868. *Cf. Geometrie der Lage*, zweite Abteilung, p. 125.

‡ Lie discusses only the case of a reciprocation, and thus the line-sphere transformation did not appear until later, as already remarked.

the special complex whose axis is IJ, *i. e.* the fundamental complex. Ambiguity in space is best avoided by regarding the transformation as a duality between two spaces r and R, such that the points of each correspond to the lines of a line complex in the other, these complexes being in R all secants of a fixed conic, and in r a general linear complex and the fundamental complex. We note here the essential characteristics of the line-sphere transformation; for the range on an arbitrary line in r must give rise to a ruled quadric containing the fixed conic, and if the latter is the imaginary circle at infinity, the quadric becomes a sphere.*

If we require that l'_0 also shall belong to the fundamental pencil, then the range on l_0 does contain p_0, and we readily see that the complex in either R or r degenerates into two special complexes, one of which is the fundamental complex.

The consideration of other special cases would not be without interest. Those enumerated may suffice in this place, however, and are moreover particularly signalled by Lie in all references to this subject.†

4. Duality of Space Defined by Two Bilinear Equations.

Writing the equation of a correlation in XZ in the form

$$(4)\quad (Ax + Bz + C)X + (A'x + B'z + C')Z + A''x + B''z + C'' = 0,$$

or also

$$(5)\quad (AX + A'Z + A'')x + (BX + B'Z + B'')z + CX + C'Z + C'' = 0,$$

and confining ourselves to *null points*, *i. e.* to real values of z and Z, then writing $x + iy$ and $X + iY$ for x and X respectively, and separating (4) into real and imaginary parts, we obtain

$$(6)\qquad (a + ia')(X + iY) + (c + ic')Z + b + ib' = 0, \text{ or}$$

$$(7)\qquad \begin{cases} aX - a'Y + cZ + b = 0 \\ a'X + aY + c'Z + b' = 0, \end{cases}$$

that is since a, a', b, b', c, c' are *linear* in x, y, z, *two bilinear equations*. Denoting space by r or R according as we represent a point by x, y, z, or X, Y, Z, we may state

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* Any point of R not on the fixed conic corresponds in r to a line of the general linear complex. But a point on that conic gives all lines in a plane through the axis of the fundamental complex. This representation of the general linear complex upon point space was first noticed by Noether, *Göttinger Nachrichten*, 1869, p. 305.

† *Cf. e. g. Mathematische Annalen*, vol. 5 (1872), p. 165; *Leipziger Berichte*, 1897, p. 728.

THEOREM 7. *Lie's representation applied to a correlation in the plane leads to a duality in space defined by two bilinear equations.*

Plücker discussed the duality in space defined by one *aequatio directrix*, but it remained for Lie to extend this notion to two equations, and the discovery came about precisely in the manner outlined above.*

The discussion of (7) is very simple, and the properties of the transformation deduced above are very easily established. The line $a = 0$, $a' = 0$ is L_0, and its points of intersection with the ruled quadric

$$\frac{b}{c} = \frac{b'}{c'}$$

are A_1 and A_2, and each of the points gives in R a plane $Z =$ const. Furthermore, the point I in R corresponds to the *plane* $a - ia' = 0$, *i. e.* the plane $A_1 A_2 J$ in r, etc.

Turning now to special cases, suppose p_0 in r, *i. e.* the point at infinity on XX', gives a line through that point itself. For this it is necessary and sufficient that $A = 0$. The line $a = 0$, $a' = 0$, now becomes the line at infinity in XY, and the singular tetrahedron reduces to IJ and two planes parallel to XY.

Suppose, however, that p_0 in r gives a real line l_0 through it; then from (5), A'/A'' must be *real*. And as we may take, without loss of generality, the line at infinity in XZ for l_0, A' becomes zero and (4), (5), and (7) reduce to

$$(8) \qquad (Bz + C)X + (B'z + C')Z + A''x + B''z + C'' = 0$$

$$(9) \qquad A''x + (BX + B'Z + B'')z + CX + C'Z + C'' = 0,$$

$$(10) \qquad \begin{cases} (a_0 z + a_1)X - (a'_0 z + a'_1)Y + (c_0 z + c_1)Z + b = 0 \\ (a'_0 z + a'_1)X + (a_0 z + a_1)Y + (c'_0 z + c'_1)Z + b' = 0, \end{cases}$$

where b and b' are still linear in x, y, z.

Taking now any line in r, $x = mz + p$, $y = nz + p$, substituting for x and y in (10), and eliminating z, we obtain a quadric whose trace upon the plane at infinity is readily found to be given by

$$(11) \qquad \begin{gathered} (a_1 a'_0)(X^2 + Y^2) + (c_1 c'_0)Z^2 + [(a_0 c_1) + (a'_0 c'_1)]\,YZ + \\ [(a'_0 c_1) + (a_1 c'_0)]'XZ = 0, \end{gathered}$$

* *Cf. Mathematische Annalen*, vol. 5 (1872), p. 143–157.

in which the parentheses denote determinants. This conic obviously passes through the circular points I and J.

The constants in (8) may be specialized with no loss of generality so that

$$B = i,\ C = -1,\ B' = -i,\ C' = -1,\ A'' = -1,\ B'' = C'' = 0.$$

In fact, this amounts to choosing (0, 0, 0), (0, 0, 1), and (1, 0, 0) in R to represent in r the axis ZZ' and the two minimum lines in xz passing through $(-1, 0, 0)$. We thus obtain from (10) the equations

$$(12)\qquad \begin{cases} X + zY + Z + x = 0, \\ zX - Y - zZ - y = 0, \end{cases}$$

while the conic (11) becomes

$$(13)\qquad X^2 + Y^2 - Z^2 = 0.$$

To obtain then from (12) the line-sphere transformation it suffices to replace Z by iZ, and these become

$$(14)\qquad \begin{cases} X + iZ + x + zY = 0, \\ z(X + iZ) - y - Y = 0. \end{cases}$$

These equations agree essentially with those adopted by Lie* by merely interchanging Z and Y.

From the equations (9) we see that the point at infinity on XX' in R gives in r an imaginary line in XZ when B/C is not real. The discussion of the previous section shows then that the line complex in r degenerates into a general linear complex and the special complex whose axis is IJ. We may readily verify this from the equations (12), for solving these for x and y, we have

$$(15)\qquad \begin{cases} x = -Yz - (X + Z), \\ y = (X - Z)z - Y. \end{cases}$$

Comparing these with the equations $x = rz + \rho$, $y = sz + \sigma$ as written by Plücker, we find

$$r = -Y,\quad \rho = -(X + Z),\quad s = X - Z,\quad \sigma = -Y,$$

* A detailed study of the line-sphere transformation is given in Lie-Scheffers, *Berührungstransformationen*, chap. 10, p. 411.

and for the remaining line-coordinate,*

$$\eta \equiv r\sigma - \rho s = X^2 + Y^2 - Z^2.$$

Then any point (X, Y, Z) not at infinity on the cone $X^2 + Y^2 - Z^2 = 0$ corresponds to a unique line of the general linear complex

$$r - \sigma = 0,$$

but the points excluded lead to the special linear complex

$$\eta = 0.$$

Finally, if the ratio B/C in (8) is real, $a_1 a_0' - a_0 a_1' = 0$, and the conic defined by (11) and the plane at infinity degenerates into two lines one of which lies in the plane XY; *i. e.* the line complex in either R or r consists of two special complexes one of which is the fundamental complex. As before, any point in R not on the degenerate conic at infinity corresponds to a line of the special complex in r whose axis is different from IJ. An example of this case is afforded by the polar reciprocation

$$(16) \qquad zZ + X + x = 0,$$

which corresponds to the transformation of space

$$(17) \qquad \begin{cases} X + x + zZ = 0, \\ Y + y = 0. \end{cases}$$

Any point not at infinity in XY or XZ corresponds to a line parallel to the plane XZ.

Lie remarks that the transformation (17) is identical with that made use of by Euler and Ampere in the theory of partial differential equations of the first order, viz.

$$X = p, \quad Y + y = 0, \quad Z + z + px = 0, \quad p = \frac{\partial z}{\partial x},$$

and in fact (17) come from these by elimination of p.

Summing up, then, we may state

THEOREM 8. *The transformation of space established by two bilinear equations in the variables (x, y, z) and (X, Y, Z) leads to a duality of the spaces r and R in which the points of either correspond to the lines of a line complex in the other. In the general case, these complexes are general tetra-*

* *Cf.* Clebsch-Lindemann, *l. c.* p 44.

hedral complexes. The special cases (1) *when the complex in* r *degenerates into a general and a special linear complex and that in* R *into all secants of a conic, and* (2) *when the complexes in both spaces degenerate into linear complexes, present themselves naturally and are among the most important of the special cases.*

Of course, by a projective transformation in either r or R, or in both, the equations (7) assume the general bilinear form. A discussion of all special cases would seem to be not without interest.

5. Point-Line Transformation in General. In 1871 Lie attacked the problem of the determination of all algebraic transformations of space such that all points go over into the lines of a line complex, and the lines of a line complex into the points of space. He was unable, at that time, to solve the problem completely, and contented himself with the remark that all* (1, 1) transformations are given by bilinear equations. The solution was finally given in his paper Liniengeometrie und Berührungstransformationen, *Leipziger Berichte*, 1897, p. 687, and the results are recapitulated on page 740. It turns out that the only cases in addition to those defined as above are:

(1) The complex in r is a general linear, and in R a special quadratic complex consisting of all the tangents of a general quadric.

(2) Both complexes are special and consist of the tangents of developable surfaces, and one of these (or both as above) may become a special *linear* complex.

The first case arises in the simplest possible manner by using a point transformation given by Darboux† by which the secants of a conic become the tangents of a general quadric. This is done by taking the case discussed above of a general linear complex in r‡ and the special complex in R of all secants of a conic, and transforming R by Darboux's point transformation.

Examples of the second type are derived by setting up a (1, 1) correspondence between the generating planes of two developables, and then assuming a duality such that a line in one plane shall correspond to a point in the other. The duality may be established, for example, by taking the con-

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* With this exception, that the general case when both complexes are linear and special is not completely represented. *Cf.* Lie, *Mathematische Annalen*, vol. 5 (1872), p. 167, footnote.

† Darboux, *Leçons sur la théorie générale des surfaces*, vol. 3, p. 493.

‡ The fundamental complex may be omitted in the statement for it arises from the ∞ points of the fundamental conic in R.

jugate of the given line with respect to a quadric, and the point of intersection of this conjugate with the corresponding plane of the other developable. In this way is obviously established a correspondence of the points of space and the tangents to a developable such that point and tangent through it go over into a like combination.

It is to be remarked that *both* complexes are general in one case only, viz. when both are tetrahedral complexes.

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